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THE STABILITY OF THE EQUILIBRIUM OF A WING IN AN UNSTEADY FLOW†

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The stability of the equilibrium position of a wing is investigated. The wing is modelled by a heavy rigid body with a fixed point and is close in shape to a thin plate. The wing is fastened using a viscoelastic material which can be modelled by non-linear viscoelastic springs that keep the wing in a position close to horizontal. The motion of the wing is described by a system of non-linear ordinary integrodifferential equations which, using the model adopted, take account of the unsteady nature of the flow past the wing and the viscoelastic properties of the spring material. The stability of the equilibrium under persistent disturbances is analysed. This analysis is based on the use of series similar to those in the first Lyapunov method. The stability of the equilibrium for purely rotational motions of the wing about the longitudinal axis is investigated in the critical case of a single zero root of the characteristic equation. The Lyapunov constants which solve the stability problem are indicated. © 2000 Elsevier Science Ltd. All rights reserved.

1. THE STABILITY OF THE EQUILIBRIUM UNDER PERSISTENT DISTURBANCES

Consider the motion of a heavy rigid body (a wing) with a fixed point, subject to the action of viscoelastic forces in the support (the wing fixing) under the influence of an unsteady flow.

We shall assume that the body is close in shape to an elongated plate with which we associate a system of coordinates $Oxyz$ with the centre at the fixed point O and axes directed along the principal axes of inertia of the body for the point O. The Oz axis coincides with the major axis of the ellipsoid of inertia, the Oy axis concludes with the miner axis and the Ox axis completes the system of coordinates.

We introduce a right-hand, rectangular system of coordinates $Ox_1y_1z_1$ with the Oy_1 axis directed along the ascending vertical and the Ox_1 axis directed opposite to the velocity vector of the unperturbed free stream. Small perturbations are superimposed on this stream.

We specify the position of the system of coordinates $Oxyz$ relative to $Ox_1y_1z_1$ by the aircraft angles: ψ (the yaw angle) is the angle of rotation of the trihedron $Oxyz$ about the Oy_1 axis from the position when the two systems of coordinates are coincident, ϑ (the pitch angle) is the angle of rotation about the Oz_1 axis in its new position Oz' after the first rotation and φ (the bank angle) is the angle of rotation about the Ox axis in its position after the second rotation.

We denote the projections of the unit vector of the Oy_1 axis by β_1, β_2 and β_3, and also the projections of the unit vector of the γ_1, γ_2, γ_3 axis onto the axes of the system of coordinates $Oxyz$ by Oz_1 and then change from the angular variables ϑ, ψ, φ, to the direction cosines β_1, β_3, γ_1 by putting

$$\sin\vartheta = \beta_1, \quad \sin\psi = -\frac{\gamma_1}{\sqrt{1-\beta_1^2}}, \quad \sin\varphi = -\frac{\beta_3}{\sqrt{1-\beta_1^2}} \tag{1.1}$$

Suppose A_i ($i = 1, 2, 3$) are the principal moments of inertia of the body for point O, ω_i are the projections of the instantaneous angular velocity vector for the rotation of the body onto the axes of the fixed system of coordinates, x_0, y_0, z_0 are the coordinates of the centre of mass of the body in the fixed axes and L_i, M_i are, respectively, the moments of the viscoelastic and aerodynamic forces with respect to the moving axes. We write the equations of motion in the form (a derivative with respect to the time t is denoted by a dot)

$$\begin{gathered} A_1\dot\omega_1 + (A_3 - A_2)\omega_2\omega_3 = mg(z_0\beta_2 - y_0\beta_3) + L_1 + M_1 \quad (1\ 2\ 3) \\ \dot\beta_1 = \omega_3\beta_2 - \omega_2\beta_3, \quad \dot\beta_3 = \omega_2\beta_1 - \omega_1\beta_2, \quad \dot\gamma_1 = \omega_3\gamma_2 - \omega_2\gamma_3 \end{gathered} \tag{1.2}$$

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We now express the quantities $\beta_2, \gamma_2, \gamma_3$ appearing in Eqs (1.2) in terms of $\beta_1, \beta_3, \gamma_1$ on the basis of geometrical considerations relating the direction cosines β_i and γ_k $(i, k = 1, 2, 3)$. We then have

$$\beta_2 = 1 + B_2, \quad \gamma_2 = -\beta_3 + \Gamma_2, \quad \gamma_3 = 1 + \Gamma_3$$

where $\beta_2, \Gamma_2, \Gamma_3$ are power series of $\beta_1, \beta_3, \gamma_1$.

We shall assume that the wing is kept in a position close to horizontal by means of viscoelastic springs (which, generally speaking, are non-linear) and, taking account of the Volterra–Fréchet representation [1, 2], we write the moments, with respect to the coordinate axes Ox, Oy, Oz, of the viscoelastic forces acting on the body in the following form

$$L_1 = l_1(\beta_3 + \beta_3^0) + \int_0^t L_1'(t-s)(\beta_3(s) + \beta_3^0)ds + L_1''(\beta_1, \beta_3, \gamma_1, t) \tag{1.3}$$

and the analogous expressions for L_2, L_3 after replacing by l_1, β_3, β_3^0, L_1', L_1'' and l_2, γ_1, γ_1^0, L_2', L_2'' respectively.

In (1.3), l_i = const, $\beta_1^0, \beta_3^0, \gamma_1^0$, = const, and L_i' are exponentially decreasing kernels (which characterize the viscoelastic properties of the material)

$$|L_i'(t)| \leqslant C_1 \exp(-\alpha_1 t), \quad \alpha_1, \ C_1 = \text{const} > 0 \tag{1.4}$$

and the non-linear terms $L_i''(\beta_1, \beta_3, \gamma_1, t)$ are functionals which include Volterra–Fréchet series (without the linear terms) and contain multiple integrals which are assumed to have exponentially decreasing kernels. The linear terms in (1.3) correspond to small rotations about the coordinate axes and follow from the kinematic equations which associate small rotations about the coordinate axes Ox, Oy, Oz with the angles ϑ, ψ, φ and, consequently, according to (1.1), with β_3, γ_1, β_1.

It can also be assumed that formulae (1.3) describe all the relaxation processes in the material of the wing support (the spring material).

Taking into account the unsteady nature of the flow past the wing, we shall take the moments of the aerodynamic forces in the form [3–5]

$$\begin{aligned}
M_i &= m_{i0} + m_{i1}\alpha + m_{i2}\beta + \sum_{j=1,2,3} m_{ij+2}\omega_j + \sum_{j=1,2,3} \int_0^t I_{ij}(t-s)\dot\omega_j(s)ds + \\
&+ \int_0^t J_{i1}(t-s)\dot\alpha(s)ds + \int_0^t J_{i2}(t-s)\dot\beta(s)ds + J_{i1}(t)\alpha(0) + J_{i2}(t)\beta(0) + \\
&+ \sum_{j=1,2,3} I_{ij}(t)\omega_j(0) + m_i(t) + M_i''
\end{aligned} \tag{1.5}$$

$$m_{ik} = \text{const}, \quad i = 1,2,3; \quad k = 1,2; \quad m_{30} = 0$$

where α is the angle of attack, β is the glancing angle, the integral kernels $l_{ij}(t), J_{ik}(t)$ and are continuously differentiable and the moments $m_i(t)$ due to small perturbations in the stream velocity are continuous, bounded functions which satisfy the estimates

$$\begin{aligned}
&|I_{ik}(t)|, \ |J_{ik}(t)|, \ |\dot I_{ik}(t)|, \ |\dot J_{ik}(t)|, \ |m_i(t)| \leqslant C_2 \exp(-\alpha_2 t) \\
&\alpha_2, \ C_2 = \text{const} > 0
\end{aligned} \tag{1.6}$$

The non-linear integral terms appearing in M_i'' only contain integral kernels which tend to zero exponentially when $t \to +\infty$. We shall assume that

$$\alpha = \vartheta + \alpha_0, \quad \beta = \psi, \quad \alpha_0 = \text{const}$$

or that

$$\alpha = \beta_1 + \alpha_0, \quad \beta = -\gamma_1 \tag{1.7}$$

It can be assumed that the constant and integral kernels in (1.5) are initially calculated for relations of the form of (1.7) so we shall assume that, in formulae (1.5), α and β are expressed in terms of β_1,

γ_1 as given by (1.7). In this case, it can also be assumed that the non-linear terms in the dependences of α and β on β_1^0, β_3^0 γ_1^0 are attributed to the terms M_i'' in (1.5).

We shall assume that the constants β_1^0, β_3^0 γ_1^0 in (1.3) for specified l_i are chosen so that the elastic forces maintain the resting wing in a position for which $\varphi = \psi = \vartheta = 0$ when there are no integral (viscoelastic) and non-linear terms in (1.3) and no integral terms in the aerodynamic moments (1.5).

We shall put

$$\omega_i = x_i, \quad i = 1,2,3; \quad \beta_1 = x_4, \quad \gamma_1 = x_5, \quad \beta_3 = x_6$$

After transforming the integral terms in (1.5), we write Eqs (1.2) in the form

$$\begin{aligned}
&\dot{x}' = Ax + \int_0^t K(t-s)x(s)ds + \phi(t) + X'(x) + Y'(x)\\
&\dot{x}'' = Jx' + X''(x)\\
&x = \mathrm{col}(x_1, \ldots, x_6), \quad x' = \mathrm{col}(x_1, x_2, x_3), \quad x'' = \mathrm{col}(x_4, x_5, x_6), \quad \phi(t) = \mathrm{col}(\phi_1, \phi_2, \phi_3)\\
&X' = \mathrm{col}(X_1', X_2', X_3'), \quad X'' = \mathrm{col}(X_1'', X_2'', X_3''), \quad Y' = \mathrm{col}(Y_1', Y_2', Y_3')
\end{aligned} \tag{1.8}$$

where $A = \|a_{ij}\|$ and $J = \|J_{ik}\|$—are constant 3×6 and 3×3 matrices respectively, the non-zero elements of matrix J constitute the diagonal $J_{13} = 1, J_{22} = -1, J_{31} = -1$; $K(t) = \|K_{ij}(t)\|$—is a continuous matrix, the elements of which, by virtue of (1.4) and (1.6), tend exponentially to zero when $t \to +\infty$, the functions X', X'' contain the non-linear terms of the corresponding equations (1.2), the functional Y' contains the non-linear terms of the integral representations (1.3) and (1.5) and $\phi(t)$ is a continuous function which tends to zero as $t \to +\infty$. For example, for the first equation of (1.8) the elements of the matrices A, $K(t)$ and the function $\phi_1(t)$ have the following values

$$\begin{aligned}
&a_{1j} = \frac{m_{1j+2} + l_{1j}(0)}{A_1}\\
&a_{14} = \frac{m_{11} + J_{11}(0)}{A_1}, \quad a_{15} = -\frac{m_{12} + J_{12}(0)}{A_1}, \quad a_{16} = \frac{l_1 - mgy_0}{A_1}\\
&K_{1j}(t) = \frac{\dot{l}_{1j}(t)}{A_1}, \quad K_{14}(t) = \frac{\dot{J}_{11}(t)}{A_1}, \quad K_{15}(t) = -\frac{\dot{J}_{12}(t)}{A_1}, \quad K_{16}(t) = \frac{L_1'(t)}{A_1}\\
&\phi_1(t) = \frac{1}{A_1}\left(m_1(t) + \beta_3^0 \int_{+\infty}^t L_1'(s)ds + \alpha_0 J_{11}(t)\right), \quad j = 1,2,3
\end{aligned} \tag{1.9}$$

The expressions for a_{ps}, $K_{ps}(t)$, $\phi_p(t)$ ($p = 2, 3$) are similar to (1.9) and easily follow from (1.2). The functions $X_i'(x)$, $X_i''(x)$ in Eqs (1.8) are defined by the formula

$$X_1' = ((A_2 - A_3)x_2x_3 + mgz_0B_2(x''))/A_1, \quad X_1'' = -x_2x_6 + x_3B_2(x'') \tag{1.10}$$

and by analogous expressions when $i = 2, 3$.

The characteristic equation

$$\det\left(A_2' + K_2^*(\lambda) - \lambda^2 J^T + \lambda\left(A_1' + K_1^*(\lambda)\right)J^T\right) = 0 \tag{1.11}$$

in which

$$A_1' = \|a_{ik}\|, \quad A_2' = \|a_{ik+3}\|, \quad K_1^*(\lambda) = \|K_{ik}^*(\lambda)\|, \quad K_2^*(\lambda) = \|K_{ik+3}^*(\lambda)\|$$

$$K_{ij}^*(\lambda) = \int_0^{+\infty} \exp(-\lambda s)K_{ij}(s)ds, \quad i,k = 1,2,3; \quad j = 1, \ldots, 6$$

corresponds to Eqs (1.8) and (1.9).

We shall assume that the perturbations associated with the viscoelastic properties of the material and the unsteady nature of the flow past the wing (in particular, gusts of wind) are small, that is, in Eqs

(1.8)–(1.10), we shall assume that

$$L_i'(t) = \mu\tilde{L}_i(t), \quad I_{ij}(t) = \mu\tilde{I}_{ij}(t), \quad J_{ij}(t) = \mu\tilde{J}_{ij}(t), \quad m_i(t) = \mu\tilde{m}_i(t)$$

$$\left|\tilde{L}_{ij}(t)\right|, \; \left|\tilde{J}_{ij}'(t)\right|, \; \left|\tilde{m}_i(t)\right| < \text{const when } t \geqslant 0$$

where μ $(0 < \mu \ll 1)$ is a small parameter. Then, in Eqs (1.8), the function $\phi(t)$ will contain the parameter μ as a factor.

Consider Eqs (1.8)–(1.10) in which the functions $\phi_i(t)$ are assumed to be persistent disturbances. Suppose all the roots λ_s' of Eqs (1.11) in the domain of its definition, which is given, by virtue of (1.4) and (1.6), by the inequality Re $\lambda \geqslant -\min(\alpha_1, \alpha_2)$, are such that Re $\lambda_s' \leqslant 1 < 0$ for a certain l. Then, the resolvent of the linearized equation (1.8) tends to zero and admits of an exponential estimate. Consequently, the theorem in [6] on stability under persistent disturbances is applicable and, according to this theorem, the general solution of Eqs (1.8) in a certain neighbourhood of the point $x_i = 0$, $(i = 1, \ldots, 6)$ is represented by absolutely convergent power series in the initial values $x_{0i} = x_i(0)$ of the variables x_i and of the parameter μ with coefficients which tend exponentially to zero when $t \to +\infty$. The point $x_i = 0$ is stable under persistent disturbances. Every solution of Eqs (1.8), for which $[x_{0i}] < \delta$ $(i = 1, \ldots, 6)$, $\mu < \delta$ for a certain $\delta > 0$, tends asymptotically to zero when $t \to +\infty$.

2. DETERMINATION OF THE LYAPUNOV CONSTANT G_2 IN THE CRITICAL CASE.

We shall now analyse the stability in the critical case of a single zero root and consider rotational motions of the wing which enables us to obtain an explicit formula for the Lyapunov constant. We shall follow the scheme for calculating these constants pointed out earlier in [7, 8].

Suppose the wing rotates around the longitudinal axis of its ellipsoid of inertia for the point O. Then, subject to the conditions that $\alpha_0 = 0$, only elastic forces act in the supports and the moment of the aerodynamic forces does not contain terms associated with perturbations of the free stream, Eqs (1.8) reduce to the following equations

$$\begin{aligned} \dot{x}_3 &= \sum_{j=3,4}\left(a_{3j}x_j + \int_0^t K_{3j}(t-s)x_j(s)ds\right) + F_3 \\ \dot{x}_4 &= x_3 + F_4 \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &a_{33} = \frac{1}{A_3}(m_{35} + I_{33}), \quad a_{34} = \frac{1}{A_3}(mgy_0 + l_3 + m_{31} + J_{31}(0)) \\ &K_{33}(t) = \frac{\dot{I}_{33}(t)}{A_3}, \quad K_{34}(t) = \frac{\dot{J}_{31}(t)}{A_3} \\ &F_3 = \frac{mgx_0}{2A_3}x_4^2 + F' + F'' + \ldots, \quad F_4 = -\frac{1}{2}x_3x_4^2 + \ldots \end{aligned} \tag{2.2}$$

The functions $K_{3j}(s)$ possess the property $K_{3j}(t) \in e_1(-\gamma)$ for certain $\gamma > 0$, that is, the satisfy the inequality

$$|K_{3j}(t)| < C\exp(-\gamma t), \qquad C = \text{const} > 0$$

The quadratic integral terms F' and the third-degree terms F'', which will be described in detail in section 3 when determining the constant F' are separated out in Eqs (2.1) and (2.2). It is assumed that the corresponding terms are included in the expressions for the moments of the aerodynamic forces. After transforming the integral terms, the functional g_3, on which the Lyapunov constant g_2 depends, is represented in the form

$$F' = F'_{33} + F'_{34} + F'_{44}$$

$$F'_{sp} = \int_0^t\int_0^t K'_{sp}(t-s_1, t-s_2)x_s(s_1)x_p(s_2)ds_1ds_2, \quad s, p = 3, 4 \tag{2.3}$$

where the integral kernels $K'_{sp}(s_1, s_2)$ are assumed to be continuous when $0 \leqslant s_i \leqslant t < +\infty$ ($i = 1, 2$) and possess the property

$$\left|K'_{sp}(s_1, s_2)\right| \leqslant C\exp(-\alpha'_1 s_1 - \alpha'_2 s_2), \quad C, \ \alpha'_1, \ \alpha'_2 = \text{const} > 0$$

We shall assume that the conditions of Theorem 1.1, formulated in [9], are satisfied. This theorem also holds when there are integral terms of the form of (2.3) with kernels of the type considered below on the right-hand sides of the equations, as was noted in [8].

Suppose the characteristic equation for system (2.1)

$$\Phi(\lambda) \equiv \lambda^2 - \lambda\left(a_{33} + K^*_{33}(\lambda)\right) - a_{34} - K^*_{34}(\lambda) = 0 \tag{2.4}$$

has a finite number of roots λ'_i ($i = 1, \ldots, L$) within the domain of its definition, that $\lambda'_L = 0$ and $\lambda'_{L-1} < 0$, and the remaining roots λ'_s are numbered in increasing order of their real parts and such that $\operatorname{Re}\lambda'_s > -\gamma$ ($s = 1, \ldots, L-2$, $0 < \gamma < \alpha_2$). Then, the condition

$$a_{34} = -K^*_{34}(0)$$

must be satisfied in the case of Eqs (2.1).

The above equality which, according to relations (2.2), can be rewritten in the form

$$mgy_0 - l + m_{31} = 0, \quad l = -l_3 > 0 \tag{2.5}$$

imposes a condition on the location of the centre of pressure of the wing. Suppose c is the distance from the Oz_1 axis to the centre of pressure C (when $\alpha = 0$) and R is the magnitude of the normal component of the principal aerodynamic force vector applied at this centre (Fig. 1). Then, by confining ourselves to the linear steady state part of the force R, its moment can be written in the form

$$M = m_{31}\alpha \equiv c\varkappa(\rho V^2/2)\alpha, \quad \varkappa = \text{const}$$

where $pV^2/2$ is the pressure head. Consequently, relation (2.5) will be satisfied if

$$c = (l - mgy_0)(\varkappa\rho V^2/2)^{-1}$$

We rewrite the general solution of the linearized equation (2.1) in the form [10]

$$x(t) = \operatorname{col}(x_3(t), x_4(t)) =$$

$$= X(t)x(0) \equiv \left(\frac{\Phi_1(0)}{\Phi'(0)} + \frac{\Phi_1(\lambda_1)}{\Phi'(\lambda_1)}\exp(\lambda_1 t) + \hat{X}(t)\right)x(0) \tag{2.6}$$

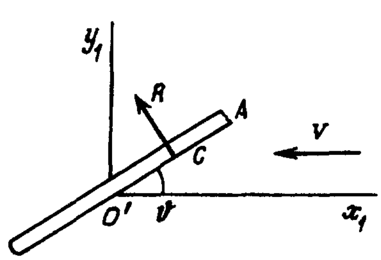

Fig. 1.

where

$$\lambda_1 = \lambda'_{L-1}, \quad X(t) = \|x_{ij}(t)\|, \quad X(0) = E_2, \quad \hat{X}(t) \in e_1(-\lambda_0)(\lambda_0 > -\lambda_1)$$

$$\Phi_1(\lambda) = \begin{Vmatrix} \lambda & -K_{34}^*(0) + K_{34}^*(\lambda) \\ 1 & \lambda - a_{33} - K_{33}^*(\lambda) \end{Vmatrix}, \quad \Phi'(\lambda) = \frac{d\Phi(\lambda)}{d\lambda}$$

Suppose $X'(t) = (x'_{ij}(t))(i,j = 1,2)$ is the Lyapunov normal, fundamental matrix of linearized system (2.1). We now introduce a matrix $Y'(t) = \|y'_{ij}(t)\|$ $(i,j = 1,2)$ which is such that $Y'(t)X'(t) = E_2$ and, following the scheme for calculations the Lyapunov constant g_3 [7, 8], we make the replacement of variable x_3, by putting

$$y = y'_{11}(t)x_3 + y'_{12}(t)x_4 \tag{2.7}$$

subject to the condition that $y'_{11}(t) \neq 0$ when $t \geqslant 0$. As a result, we obtain the equations

$$\dot{y} = y'_{11}(t)F_3(x_3, x_4) + y'_{12}(t)F_4(x_3, x_4) + \\ +\int_0^t [\varphi_1(t,s)F_3(x_3(s), x_4(s)) + \varphi_2(t,s)F_4(x_3(s), x_4(s))]ds \tag{2.8}$$

$$\varphi_j(t,s) = \frac{\partial}{\partial t}\left(y'_{11}(t)x_{1j}(t-s) + y'_{12}(t)x_{2j}(t-s)\right)$$

$$\dot{x}_4 = \left(y - y'_{12}(t)x_4\right)/\, y'_{11}(t) + F_4(x_3, x_4) \tag{2.9}$$

in which the quantity x_3 is expressed in terms of y and x_4 according to relation (2.7).

Next, we transform Eq (2.9) using the substitution

$$z = \exp(\lambda_1 t)x_4 \,/\, x'_{22}(t) \tag{2.10}$$

As a result, Eq (2.9) for the non-critical variable becomes

$$\dot{z} = \lambda_1 z + \frac{\exp(\lambda_1 t)}{x'_{22}(t)}\left(\frac{y}{y'_{11}(t)} + F'_4(y,z,t)\right) \tag{2.11}$$

with bounded coefficients when $t \geqslant 0$. In Eq (2.11). $F'_4(y,z,t)$ are the non-linear terms $F_4(x_3, x_4)$ which have been transformed to the variables y and z.

We now eliminate the term which is linear in y from Eq (2.11) and, for this purpose, we carry out the transformation of the variable z

$$v = z + u_1(t)y, \quad u_1(t) = -\exp(\lambda_1 t)\int_0^t \frac{ds}{x'_{22}(s)y'_{11}(s)} \tag{2.12}$$

in which the continuous, bounded function $u_1(t)$ can be represented as

$$u_1(t) = u_0 + \tilde{u}_1(t), \quad u_0 = -\frac{\lambda_1\Phi'(\lambda_1)}{l_1^2\Phi'(0)}, \quad l_1 = \lambda_1 - K_{33}^*(\lambda_1) + K_{33}^*(0) \tag{2.13}$$

where the function $\bar{u}_1(t) \in e_1(-\alpha_0)$ for a certain $\alpha_0 > 0$.

Taking account of relations (2.10) and (2.12), we now determine (sport from an additive function which tends to zero as $t \to +\infty$) the coefficient of y^2 in Eq (2.8) which we shall denote by $g'_2(t)$. We have

$$g'_2(t) = \left(y'_{11}(t) + \int_0^t \varphi_1(t,s)ds\right)\left(\frac{mgx_0}{2A_3}(x'_{22}(t)\exp(-\lambda_1 t))^2 u_1^2(t) + \tilde{F}(t)\right) \tag{2.14}$$

where $\bar{F}(t)$ is a function which is generated by the integral terms F' in (2.3) and will be more precisely defined below.

Starting out from relation (2.14), we obtain the Lyapunov constant $g_2 = \lim g_2'(t)$ when $t \to +\infty$.

We calculate the constants φ_1^0, y_{11}^0, appearing in the representation of the functions

$$\int_0^t \varphi_j(t,s)ds = \varphi_j^0 + \tilde{\varphi}_j(t), \quad y_{1j}'(t) = y_{1j}^0 + \tilde{y}_{1j}(t)$$
$$\tilde{\varphi}_j(t), \tilde{y}_{1j}(t) \in e_1(-\alpha_0), \quad j = 1,2 \tag{2.15}$$

On the basis of formulae (2.8) and (2.6) and taking account of the fact that $x_{11}(0) = 1, x_{21}(0) = 0$, we obtain

$$\varphi_1^0 = \frac{\Phi'(0)}{\lambda_1^2}\left(l_1 + \frac{\lambda_1^2}{\Phi'(0)}\right), \quad y_{11}^0 = -\frac{l_1}{\lambda_1^2}\Phi'(0) \tag{2.16}$$

Next, we make the following assumption regarding the structure of the integral kernels in formulae (2.3)

$$K_{sp}'(t-s_1, t-s_2) = K_{sp}^{(1)}(t-s_1)K_{sp}^{(2)}(t-s_2)$$

Suppose that $F_{sp}^0 = \lim F_{sp}'$ when $t \to +\infty$ in (2.3). Carrying out the corresponding calculations, we obtain

$$F_{sp}^0 = \frac{k_{sp}^{(1)}k_{sp}^{(2)}}{(\Phi'(0))^2}(l_1 - \lambda_1)^{8-s-p}\left(-\frac{\lambda_1}{l_1}\right)^{10-s-p}$$
$$k_{sp}^{(i)} = \int_0^\infty K_{sp}^{(i)}(s)ds, \quad i = 1,2; \quad s,p = 3,4 \tag{2.17}$$

Finally, by relations (2.14)–(2.17), we have the expression

$$g_2 = \frac{mgx_0}{2A_3l_1^2}\frac{\lambda_1^2}{(\Phi'(0))^2} + F_{33}^0 + F_{34}^0 + F_{44}^0 \tag{2.18}$$

for the Lyapunov constant.

Hence, on the basis of the theorem on instability in [8, 9], if $g_2 \neq 0$, the equilibrium position under consideration is unstable.

In the special case, when there are no quadratic integral terms (2.3) in the aerodynamic moments, formula (2.18) leads to the instability condition $x_0 \neq 0$. It is of interest to investigate the case when $g_2 = 0$. Then, the stability of the equilibrium position depends on the sign of the constant g_3, which is determined by the terms of up to the third order of magnitude inclusive on the right-hand sides of the equations.

3. DETERMINATION OF THE LYAPUNOV CONSTANT G_3

Suppose $g_2 = 0$. We shall discuss the steady part of the moment of the aerodynamic forces. We will assume that the wing is rectangular and we will denote the distance from the leading edge of the wing A to the centre of pressure C by S and the chord of the wing by d. The cross-section of the wing is the plane orthogonal to the Oz_1 axis and which passes through C, showed schematically in the figure, where O' is the point of intersection of Oz_1 with this plane. On the basis of results of investigations carried out previously [11] in the domain of small changes in the angle of attack α, the relations

$$S = d(s_0 + s_2\alpha^2), \quad R = r_1\alpha - r_3\alpha^3$$
$$(s_0, s_2 r_1, r_3 = \text{const} > 0)$$

can be taken as an approximation for S and the magnitude of the normal pressure R. For the steady part M' of the moment of the aerodynamic forces, which depend on α, this gives the expression

$$M' = m'_{31}\alpha + m'\alpha^3 \tag{3.1}$$
$$m'_{31} = ds_0 r_1 > 0,\ \ m' = -(cr_3 + ds_2 r_1),\ \ c = O'A - ds_0$$

up to terms in third-order infinitesimals.

Taking account of representation (3.1), we shall consider the following model of the moment of the aerodynamic forces acing on the wing assuming, for simplicity, that unsteady flow past the wing is solely taken into account by terms of the first order. We shall assume that the moment M_3 of the aerodynamic forces is specified, by analogy with formula (1.2), obtained after reduction, by the following relation

$$M_3 / A_3 = m'_{31}\alpha + m'_{35}\omega_3 + m'\alpha^3 + m''\omega_3^3 +$$
$$+\int_0^t K'_{33}(t-s)\omega_3(s)ds + \int_0^t K'_{31}(t-s)\alpha^{(1)}(s)ds \tag{3.2}$$

in which the dissipative term $m'_{35}\omega_3$ is added and also a term containing ω_3^3. The angle of attack α, which is calculated for the leading edge of the wing as the angle between the relative velocity vector of the free stream and the plane of the wing, is given by the formula (V is the modulus of the absolute velocity of the stream)

$$\operatorname{tg}\alpha = (\beta_1 - (a/V)\omega_3)/\sqrt{1-\beta_1^2}$$

which gives the expansion

$$\alpha = \alpha^{(1)} + \alpha^{(3)} + \ldots$$
$$\alpha^{(1)}(\omega_3,\beta_1) = \beta_1 - \frac{a}{V}\omega_3,\ \ \alpha^{(3)}(\omega_3,\beta_3) = \frac{1}{2}\beta_1^2\alpha^{(1)} - \frac{1}{3}(\alpha^{(1)})^3 \tag{3.3}$$

As a result, in the case in question, we shall have Eqs (2.1) in which by (3.2),

$$a_{33} = m'_{35} - (a/V)m'_{31},\ \ a_{34} = m'_{31} + (mgz_0 - l)/A_3$$
$$K_{33}(t) = K'_{33}(t) - (a/V)K'_{31}(t),\ \ K_{34}(t) = K'_{31}(t) \tag{3.4}$$
$$F''(x_3, s_4) = m'_{31}\alpha^{(3)}(x_3,x_4) + m'(\alpha^{(1)}(x_3,x_4))^3 + m''x_3^3$$

Apart from an additive function of the class $e_1(-\alpha_0)$ ($\alpha_0 > 0$), the coefficient of the term containing y^3 in the equation for the critical variable can be written, taking account of the condition $g_2 = 0$, as follows:

$$g'_3(t) = \left(y'_{11}(t) + \int_0^t \varphi_1(t,s)ds \right) F''(\hat{x}_3(t), \hat{x}_4(t)) -$$
$$-\frac{1}{2}\left(y'_{12}(t) + \int_0^t \varphi_2(t,s)ds \right)\hat{x}_3(t)\hat{x}_4^2(t) \tag{3.5}$$

where

$$\hat{x}_3(t) = (1 + u_1(t)y'_{12}(t)x'_{22}(t)\exp(\lambda_1 t))(y'_{11}(t))^{-1}$$
$$\hat{x}_4(t) = -u_1(t)x'_{22}(t)\exp(\lambda_1 t) \tag{3.6}$$

We represent the function (3.6) in the form

$$\hat{x}_i(t) = x_i^0 + \tilde{x}_i(t),\ \ \tilde{x}_i(t) \in e_1(-\alpha_0),\ \ \alpha_0 > 0$$

and calculate the constants x_i^0 and also the constants y_{12}^0, φ_2^0 introduced in (2.15). We have

$$x_3^0 = -\frac{\lambda_1^2}{l_1\Phi'(0)}\left(1-\frac{\lambda_1}{l_1}\right),\ x_4^0 = \frac{\lambda_1}{l_1\Phi'(0)}$$
$$y_{12}^0 = \Phi'(0),\ \varphi_2^0 = -\Phi'(0) \tag{3.7}$$

On separating out the constant term from g_3', (t) we find $g_3 = F'''(x_3^0, x_4^0)$ and, from (3.4)–(3.7) and (2.16), we obtain the expression

$$g_3 = \left(\frac{\lambda_1}{l_1\Phi'(0)}\right)^3 \left\{ m_{31}'\left(1+\frac{a}{V}L\right)\left[\frac{1}{2}-\frac{1}{3}\left(1+\frac{a}{V}L\right)^2\right]+ \right.$$
$$\left. +m'\left(1+\frac{a}{V}L\right)^3 - m''L^3 \right\} \tag{3.8}$$

in which, by (3.4) and (2.4)

$$\Phi'(0) = -m_{35}' + \frac{a}{V}m_{31}' + \int_0^\infty (-K_{33}(s)ds + sK_{34}(s))ds$$

$$L = \lambda_1[1-\lambda_1(\lambda_1 - K_{33}^*(\lambda_1) + K_{33}^*(0))^{-1}]$$

If $g_3 > 0$, the equilibrium is unstable. If $g_3 < 0$ and the integral kernels in Eq (2.1) have an exponential-polynomial structure, the equilibrium is asymptotically stable [9].

When there are no integral terms which take account of the unsteady nature of the flow past the wing, we have $l_1 = \lambda_1$, $L = 0$, and formula (3.8) takes the form

$$g_3 = \frac{1}{(\Phi'(0))^3}\left(\frac{1}{6}m_{31}' + m'\right) \tag{3.9}$$

The constant g_3 in (3.9) is identical to the analogous constant determined using the theory of critical Lyapunov cases for differential equations.

On taking account of the dissipative nature of the terms $m_{35}'\omega_3$ in formula (3.2) and the interpretation of the coefficient m_{31}' given in (3.1), it can be concluded that, in (3.9), the quantity $\Phi'(0) > 0$.

It then follows from (3.9) that there is the asymptotic stability when $m' < -m_{31}'/6$.

The effect of the unsteady nature of the flow past the wing on the stability turns out to be substantial if the constant g_3 (3.9) is close to zero.

Note that the constant g_3 changes if the applied forces are changed and, in particular, if, for example, we take a quantity which is linear in ϑ, as the moment of the elastic forces unlike the moment of these forces assumed in sections 2 and 3, which is linear in $\beta_1 = \sin\vartheta$.

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